TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 11, November 1995

ON SOME SUBALGEBRAS OF $B(c_0)$ AND $B(l_1)$

F. P. CASS AND J. X. GAO

ABSTRACT. For a non-reflexive Banach space X and $w \in X^{**}$, two families of subalgebras of $B(X)$, $\Gamma_w = \{T \in B(X) \mid T^{**}w = kw \text{ for some } k \in \mathbb{C}\}$, and $\Omega_w = \{T \in B(X) \mid T^{**}w \in w \oplus \hat{X}\}$ for $w \in X^{**} \setminus \hat{X}$ with $\Omega_w = B(X)$ for $w \in \hat{X}$, were defined originally by Wilansky. We consider $X = c_0$ and $X = l_1$ and investigate relationships between the subalgebras for different $w \in X^{**}$. We prove in the case of c_0 that, for $w \in X^{**} \setminus \hat{X}$, all Γ_w's are isomorphic and all Ω_w's are isomorphic. For $X = l_1$, where it is known that not all Γ_w's are isomorphic and not all Ω_w's are isomorphic, we show, surprisingly, that subalgebras associated with a Dirac measure on $\beta\mathbb{N} \setminus \mathbb{N}$, regarded as a functional on l_1^*, are isomorphic to those associated with some Banach limit (i.e., a translation invariant extended limit). We also obtain a representation for the operators in the subalgebras $\{\bigcap \Gamma_f \mid f \text{ is a Banach limit}\}$ and $\{\bigcap \Omega_f \mid f \text{ is a Banach limit}\}$ of $B(l_1)$.

1. INTRODUCTION

Let X be a non-reflexive Banach space over $\mathbb{C}$ and denote by $B(X)$ the Banach algebra of all bounded linear operators from X to itself. For $x \in X$, $\hat{x}$ is the image of x under the natural embedding of X into X^{**}. For $w \in X^{**}$ let $\langle w \rangle$ denote the one dimensional subspace of X^{**} generated by w. Following Wilansky [9], but using different notation, we define

$$\Gamma_w = \{T \in B(X) \mid T^{**}w \in \langle w \rangle\}, \quad \text{for } w \in X^{**},$$
$$\Omega_w = \{T \in B(X) \mid T^{**}w \in w \oplus \hat{X}\}, \quad \text{for } w \in X^{**} \setminus \hat{X}.$$

Here the definition of Ω_w can be extended to $w \in \hat{X}$ by setting $\Omega_w = B(X)$.

For $w \in X^{**} \setminus \hat{X}$, we define $\rho = \rho_w : \Omega_w \to \mathbb{C}$ by the equation

$$T^{**}w = \rho_w(T)w + \hat{x}, \quad T \in \Omega_w.$$

It is clear that for $T \in \Gamma_w$, $T^{**}(w) = \rho_w(T)w$.

Wilansky [9] proved the following properties of these algebras:

Theorem A. *For $w \in X^{**}$, Γ_w is a closed subalgebra of $B(X)$. For $w \in X^{**} \setminus \hat{X}$, Ω_w is a closed subalgebra of $B(X)$ and ρ_w is a nonzero continuous scalar homomorphism on Ω_w.*

Received by the editors March 29, 1994 and, in revised form, December 27, 1994.
1991 *Mathematics Subject Classification.* Primary 47D30, 46A45; Secondary 46B10.
Key words and phrases. Subalgebras, operators on Banach spaces, algebraic isomorphism, Banach limits, Dirac measures, Stone-Čech compactification.

We shall use the following non-reflexive Banach spaces of sequences of complex numbers.

$$c_0 = \left\{ \{x_k\}_{k=1}^{\infty} \mid \lim_{k\to\infty} x_k = 0,\ \|\{x_k\}\| = \|\{x_k\}\|_\infty = \sup_k |x_k| \right\},$$

$$c = \left\{ \{x_k\}_{k=1}^{\infty} \mid \lim_{k\to\infty} x_k \text{ exists, } \|\{x_k\}\| = \|\{x_k\}\|_\infty \right\},$$

$$l_1 = \left\{ \{x_k\}_{k=1}^{\infty} \mid \|\{x_k\}\| = \|\{x_k\}\|_1 = \sum_k |x_k| < \infty \right\},$$

$$l_\infty = \left\{ \{x_k\}_{k=1}^{\infty} \mid \|\{x_k\}\| = \|\{x_k\}\|_\infty < \infty \right\}.$$

For a sequence $x = \{x_k\}$, we define $P_n x = x_n$ for all n.

We shall denote by e the sequence all of whose terms are 1. For a set E of non-negative integers, we write $\chi(E)$ for the sequence $\{x_k\}$ such that $x_k = 1$ for $k \in E$ and $x_k = 0$ otherwise. We write, for any fixed $n = 1, 2, \dots$, $e_n = \chi(\{n\})$.

The continuous linear functional lim on c is defined by $\lim x = \lim_{k\to\infty} x_k$. By the Hahn-Banach theorem, lim can be extended to l_∞ while preserving the norm 1. We call such an extension an extended limit.

We list the following theorem here for ease of reference. It gives us some general properties of the subalgebras Γ_w and Ω_w. The proof of it can be found in Brown, Cass and Robinson [2].

Theorem B. *Let X be a non-reflexive Banach space, and let $w_1, w_2 \in X^{**}$,*

(1) *If $w_1 = \mu w_2$ for some $\mu \neq 0$, then $\Gamma_{w_1} = \Gamma_{w_2}$. If $\langle w_1\rangle + \hat{X} = \langle w_2\rangle + \hat{X}$, then $\Omega_{w_1} = \Omega_{w_2}$.*
(2) *If $w_i \neq 0$ for $i = 1, 2$, and $w_1 \notin \langle w_2\rangle$, then $\Gamma_{w_1} \setminus \Gamma_{w_2} \neq \varnothing$ and $\Gamma_{w_2} \setminus \Gamma_{w_1} \neq \varnothing$.*
(3) *If $w_1 \neq 0$, then $\Gamma_{w_1} \neq \Omega_{w_1}$.*

Corollary. *$\Gamma_{w_1} = \Gamma_{w_2}$ if and only if there is a number $\mu \neq 0$ such that $w_1 = \mu w_2$.*

The question of whether $w_1 \oplus \hat{X} = w_2 \oplus \hat{X}$ when $\Omega_{w_1} = \Omega_{w_2}$ is not yet resolved in general. Brown and Cho [3] investigated these algebras when $X = c$ and gave a positive answer as follows. The natural embedding used is:

$$c \to c^{**} = l_\infty, \quad x = \{x_k\} \mapsto \{\lim x, x_1, x_2, \dots\}.$$

Theorem C. *Let $w \in l_\infty \setminus \hat{c}$, then $\Omega_z = \Omega_w$ if and only if $z \in (w \oplus \hat{c}) \setminus \hat{c}$.*

We shall use the following lemma to investigate isomorphisms between the two families of subalgebras Γ_w and Ω_w.

Lemma D. *Suppose X is a non-reflexive Banach space and $z, w \in X^{**}$. If there is an invertible $T \in B(X)$ such that $T^{**}z = aw$ for some $a \in \mathbb{C}$, then $\Gamma_z \cong \Gamma_w$. If, in addition, $z, w \notin \hat{X}$, then $\Omega_z \cong \Omega_w$.*

Lemma D can be proved simply by using the mapping

$$U \mapsto TUT^{-1} \qquad \text{for } U \in B(X). \tag{1}$$

Remark. It was shown in Theorem 9 of [2], Theorem 1 of [5] and Theorem 1 of [6] that all isomorphisms between Γ_w 's or between Ω_w 's are necessarily given in the form of (1).

Brown and Cho [3], in their study of $B(c)$, proved the following theorem:

Theorem E. *Let $w \in l_\infty \setminus \hat{c}$, then there exists an automorphism T in $B(c)$ such that $T^{**}w = e_1 \in l_\infty \setminus \hat{c}$. Hence for $w, z \in l_\infty \setminus \hat{c}$, $\Gamma_w \cong \Gamma_z$ and $\Omega_w \cong \Omega_z$.*

2. Subalgebras of $B(c_0)$

We now establish a result similar to Theorem E for $B(c_0)$.

Lemma 1. *Let $w \in l_\infty \setminus \hat{c}_0$, then there is an invertible $T \in B(c_0)$ such that $T^{**}w$ is a sequence of zeros and ones.*

Proof. Suppose $w = \{w_1, w_2, \dots\} \in l_\infty \setminus \hat{c}_0$, and let $\limsup_{n\to\infty} |w_n| = \delta > 0$. Let $\{w_{n_k}\}$ be a subsequence of w such that $|w_{n_k}| \geqslant \delta/2$ for all k and $\lim_{k\to\infty} |w_{n_k}| = \delta$. Write $n_0 = 0$ and let $w_{n'_k}$ be the first term w_n having the largest absolute value among $\{w_{n_{k-1}+1}, w_{n_{k-1}+2}, \dots, w_{n_k}\}$.

Let $T_k = (a_{ij}^{(k)})$ be an $(n_k - n_{k-1}) \times (n_k - n_{k-1})$ matrix with $a_{ij}^{(k)}$ defined as:

$$a_{ij}^{(k)} = \begin{cases} 1/w_{n'_k} & \text{for } i = j = n'_k - n_{k-1}; \\ -w_{n_{k-1}+i}/w_{n'_k} & \text{for } j = n'_k - n_{k-1},\ i \neq j; \\ 1 & \text{for } i = j \neq n'_k - n_{k-1}; \\ 0 & \text{otherwise.} \end{cases}$$

In matrix form we have

$$T_k = \begin{pmatrix} 1 & 0 & \cdots & -w_{n_{k-1}+1}/w_{n'_k} & \cdots & 0 \\ 0 & 1 & \cdots & -w_{n_{k-1}+2}/w_{n'_k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1/w_{n'_k} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -w_{n_k}/w_{n'_k} & \cdots & 1 \end{pmatrix}$$

and

$$T_k^{-1} = \begin{pmatrix} 1 & 0 & \cdots & w_{n_{k-1}+1} & \cdots & 0 \\ 0 & 1 & \cdots & w_{n_{k-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & w_{n'_k} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{n_k} & \cdots & 1 \end{pmatrix}.$$

Note that $\|T_k\|_\infty \leqslant \max\{2, 2/\delta\}$ and $\|T_k^{-1}\|_\infty \leqslant 1 + \|w\|_\infty$ for $k = 1, 2, \dots$.

Let

$$T = \begin{pmatrix} T_1 & 0 & 0 & \cdots \\ 0 & T_2 & 0 & \cdots \\ 0 & 0 & T_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then $T \in B(c_0)$ and is invertible, with $T^{**}w$ being a sequence of zeros and ones. □

Theorem 1. *Let $w \in l_\infty \setminus \hat{c}_0$. Then*

(a) *there is an invertible $S \in B(c_0)$ such that $S^{**}w = e$,*

(b) *for $z \in l_\infty \setminus \hat{c}_0$, $\Gamma_w \cong \Gamma_z$ and $\Omega_w \cong \Omega_z$.*

Proof. (a) By Lemma 1, we can assume that w is a sequence of zeros and ones with infinitely many ones, say $w = \chi(\{n_k\})$. Let

$$S_k = \begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

which is of order $(n_k - n_{k-1}) \times (n_k - n_{k-1})$ with $n_0 = 0$. We see that S_k is invertible, $\|S_k\|_\infty \leqslant 2$ and $\|S_k^{-1}\|_\infty \leqslant 2$. Now

$$S = \begin{pmatrix} S_1 & 0 & 0 & \cdots \\ 0 & S_2 & 0 & \cdots \\ 0 & 0 & S_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the matrix needed to map w onto e.

(b) This follows directly from (1) and Lemma D. □

Remark. By definition, $T \in B(c_0)$ is in Ω_e if and only if $T^{**}e = \rho e + \hat{x}_0$ for some $\rho \in \mathbb{C}$ and $x_0 \in c_0$. This is equivalent to saying that for all $x \in c$, $T^{**}x = \rho(\lim x)e + \widehat{x_0'}$ for some $x_0' \in c_0$, i.e., $\lim T^{**}x = \rho \lim x$ with ρ independent of $x \in c$. Thus we observe that if we consider operators in $B(c_0)$ as matrices, then Ω_e is exactly the collection of all multiplicative conservative matrices as defined in [4]. Hence Theorem 1 shows that for each $w \in l_\infty \setminus \hat{c}_0$, Ω_w is isomorphic to the algebra of all multiplicative conservative matrices. This is an algebra that has been much studied in summability theory.

The following result for c_0 is similar to Theorem C.

Theorem 2. *Let $w = \{w_n\}$ and $z = \{z_n\} \in l_\infty$.*

(a) *If $z \notin \hat{c}_0$ and $w \notin z \oplus \hat{c}_0$, then neither Ω_w nor Ω_z contains the other.*

(b) *$\Omega_z = \Omega_w$ if and only if $\langle w \rangle + \hat{c}_0 = \langle z \rangle + \hat{c}_0$.*

Proof. (a) We first show that we can always find two disjoint increasing sequences $\{n_k'\}$ and $\{n_k''\}$ of natural numbers, such that all of $w' = \lim_k w_{n_k'}$, $w'' = \lim_k w_{n_k''}$, $z' = \lim_k z_{n_k'}$ and $z'' = \lim_k z_{n_k''}$ exist but $w'z'' \neq w''z'$. Because $z \notin \hat{c}_0$, we can find $\{n_k'\}$ such that $z' = \lim_k z_{n_k'} \neq 0$. By picking a subsequence of $\{n_k'\}$ if necessary, we can assume that $\{w_{n_k'}\}$ also converges, say to w'. Because $w - (w'/z')z \notin \hat{c}_0$, there exists $\{n_k^*\}$, disjoint with $\{n_k'\}$, such that $w_{n_k^*} - (w'/z')z_{n_k^*} \to r \neq 0$. A subsequence $\{n_k''\}$ of $\{n_k^*\}$ on which both w and z converge is all we need.

Now define $T \in B(c_0)$ by

$$P_n Tx = \begin{cases} (w'' - w')x_{n_k''} + w''x_{n_k'} & \text{for } n = n_k'',\ k = 1, 2, \ldots, \\ w''x_n & \text{for all other } n, \end{cases}$$

where $x = \{x_n\} \in c_0$. Now $P_{n'_k}(T^{**}z - w''z) = 0$ for each k. Keeping in mind that $z' \neq 0$, we have $P_{n'_k}(T^{**}z - \rho z) \not\to 0$ for any $\rho \neq w''$. But

$$P_{n''_k}(T^{**}z - w''z) = (w'' - w')z_{n''_k} + w''z_{n'_k} - w''z_{n''_k} \to w''z' - w'z'' \neq 0$$

as $k \to \infty$. Hence $T \notin \Omega_z$. Also $T^{**}w - w''w \in \hat{c}_0$. Consequently $T \in \Omega_w$ and thus $T \in \Omega_w \setminus \Omega_z$.

As the conditions are in fact symmetric on z and w, we can similarly construct an operator in $\Omega_z \setminus \Omega_w$. Thus neither Ω_w nor Ω_z contains the other.

(b) The "if" part, i.e., to show that $\Omega_z = \Omega_w$ when $\langle w \rangle + \hat{c}_0 = \langle z \rangle + \hat{c}_0$, was done in (1) of Theorem B.

To show the "only if" part, in light of (a), we only have to consider the case that exactly one of w and z is in $\hat{c}_0$, say $z \in \hat{c}_0$. By definition $\Omega_z = B(c_0)$. So we only have to find an operator in $B(c_0)$ which is not in Ω_w. Suppose a subsequence $\{w_{n_k}\}$ of w satisfies $\lim_k w_{n_k} = \delta \neq 0$, then the diagonal matrix $D = \mathrm{diag}\{d_n\} \notin \Omega_w$ if we define $d_{n_{2k}} = 1$ and all other $d_n = 0$. □

3. Subalgebras of $B(l_1)$

Let $\mathbb{N}$ be the set of all positive integers and $\beta\mathbb{N}$ the Stone-Čech compactification of $\mathbb{N}$. Each $x \in l_\infty$, as a bounded continuous function on $\mathbb{N}$, can be extended to a unique continuous function on $\beta\mathbb{N}$, which we still write as x. In this way l_∞ becomes isometric in a natural way to $C(\beta\mathbb{N})$. Following [7], we identify points of $\beta\mathbb{N}$ with ultrafilters on $\mathbb{N}$. If $E \subset \mathbb{N}$ and $p \in \beta\mathbb{N}$, then $E \in p$ if and only if $p \in \overline{E}$. Here $\overline{E}$ stands for the closure of E in $\beta\mathbb{N}$. By the Riesz Representation Theorem (see, e.g., [8, p. 130]), each $f \in l_\infty^*$ corresponds to a unique regular Borel measure μ_f on $\beta\mathbb{N}$ such that $\|f\| = \|\mu_f\|$ and

$$f(x) = \int_{\beta\mathbb{N}} x \, d\mu_f \qquad \text{for } x \in l_\infty.$$

We write $M(\beta\mathbb{N})$ for the space of all regular Borel measures on $\beta\mathbb{N}$.

Let $p \in \beta\mathbb{N} \setminus \mathbb{N}$. We use the notation Ω_p and Γ_p respectively for the subalgebras Ω_{δ_p} and Γ_{δ_p}, where δ_p is the Dirac measure at p. Cass showed in [5] that if p and q are different, then $\Omega_p \neq \Omega_q$.

The following theorem combines Theorem 10 in [2] and Theorem 2 in [5].

Theorem F. *There exist pairs of points p and q in $\beta\mathbb{N} \setminus \mathbb{N}$ such that $\Gamma_p \not\supseteq \Gamma_q$ and $\Omega_p \not\supseteq \Omega_q$.*

For $x = \{x_k\}_{k=1}^\infty \in l_\infty$, the translation of x is defined by $Tx = \{x_{n+1}\}_{n=1}^\infty$. A linear functional f on l_∞ is called a *Banach limit* if it satisfies:

(i) $f(x) = \lim x$ for all $x = \{x_n\} \in c$;

(ii) $\|f\| = 1$;

(iii) f is translation invariant, i.e., $f(x) = f(Tx)$ for all $x \in l_\infty$.

Thus a Banach limit is an extended limit which is translation invariant. The set of all Banach limits is denoted by BL.

The properties (i) and (ii) above imply:

(iv) If $x = \{x_n\} \in l_\infty$ with $x_n \geqslant 0$, then $f(x) \geqslant 0$. Hence if $x = \{x_n\} \in l_\infty$ and $x_n \in \mathbb{R}$ for all n, then $\liminf_n x_n \leqslant f(x) \leqslant \limsup_n x_n$.

The following lemma is given in Bennett and Kalton [1, Lemma 1].

Lemma G. *Suppose f is a continuous linear functional on l_∞. Then f is a Banach limit if and only if the following three properties are all satisfied:*

(i) $\|f\| = 1$;
(ii) $f(e) = 1$;
(iii) $f(x) = 0$ *for all* $x \in bs$.

Here bs is the space of series with bounded partial sums.

Remark. Unlike the Dirac measures, the measures on $\beta\mathbb{N}$ associated with Banach limits have no atoms. Indeed, consider any such measure μ and any $p \in \beta\mathbb{N}$. For $n \in \mathbb{N}$, Let $E_{i,n} = \{k \in \mathbb{N} \mid k \equiv i \mod(n)\}$ where $0 \leqslant i \leqslant n-1$. Then $p \in \overline{E}_{i,n}$ for some i and $\mu(\{p\}) \leqslant \mu(\overline{E}_{i,n}) = 1/n$. Hence $\mu(\{p\}) = 0$.

The arguments given in [2] and [5] to establish Theorem F rely on cardinality considerations which amount to showing that there are too few isomorphisms available for all the algebras Γ_p or Ω_p to be isomorphic. In view of the above remark, we thought that it might be that whenever $p \in \beta\mathbb{N}\backslash\mathbb{N}$ and f is a Banach limit, then Γ_p could not be isomorphic to Γ_f nor could Ω_p be isomorphic to Ω_f. While these algebras are different as shown by Theorem 3, we found, surprisingly, that they can be isomorphic as the Corollary to Theorem 4 shows.

Theorem 3.

$$\bigcap\{\Gamma_f : f \in BL\} \setminus \bigcup\{\Omega_p : p \in \beta\mathbb{N} \setminus \mathbb{N}\} \neq \varnothing, \tag{2}$$

and

$$\bigcap\{\Gamma_p : p \in \beta\mathbb{N} \setminus \mathbb{N}\} \setminus \bigcup\{\Omega_f : f \in BL\} \neq \varnothing. \tag{3}$$

Proof. Define $A = (a_{nk}) \in B(l_1)$ (the unilateral shift operator) by $a_{n+1,n} = 1$ for $n = 1, 2, \ldots$ and $a_{nk} = 0$ otherwise. $T = A^*$ is the translation operator on l_∞, as used in the definition of Banach limit. For every Banach limit f, $A^{**}f = f$, so $A \in \Gamma_f$.

On the other hand, if $p \in \beta\mathbb{N} \setminus \mathbb{N}$, then $A \notin \Omega_p$. Indeed, assume the converse, and suppose $A^{**}\delta_p = \rho\delta_p + \hat{y}$, where $y = \{y_k\} \in l_1$. Then for $x = \{x_k\} \in l_\infty$ we have

$$\int_{\beta\mathbb{N}} Tx\, d\delta_p = (A^{**}\delta_p)(x) = \rho \int_{\beta\mathbb{N}} x\, d\delta_p + \sum_{k=1}^{\infty} y_k x_k.$$

By putting $x = e_k$ we see $y_k = 0$. Now for $E = \{2n\}_{n=1}^{\infty}$, p is either in $\overline{E}$ or $\overline{\mathbb{N} \setminus E}$. Suppose without loss of generality that $p \in \overline{E}$. Putting $x = \chi(\mathbb{N} \setminus E)$ yields: $1 = \int_{\beta\mathbb{N}} Tx\, d\delta_p = \rho \int_{\beta\mathbb{N}} x\, d\delta_p = \rho \cdot 0 = 0$, a contradiction which shows that $A \notin \Omega_p$. Thus (2) holds.

For (3), define $D = (d_{nk}) \in B(l_1)$ by putting $d_{2m,2m} = 1$ for all $m \in \mathbb{N}$ and $d_{nk} = 0$ otherwise (D is the "even-term picking" operator). Then for $p \in \beta\mathbb{N} \setminus \mathbb{N}$, depending on whether $p \in \overline{E}$ or $p \in \overline{\mathbb{N} \setminus E}$, we have either $D^{**}\delta_p = \delta_p$ or $D^{**}\delta_p = 0$. In both cases we have $D \in \Gamma_p$. But if there is a $\rho \in \mathbb{C}$ and $z = \{z_k\} \in l_1$ such that $D^{**}f = \rho f + \hat{z}$ for some Banach limit f, then with $x = e_k$ we see $z_k = 0$. Again put $x = e$ we find $\rho = 1/2$ but if $x = \chi(E)$ we find $\rho = 1$, a contradiction which shows $D \notin \Omega_f$. Hence (3) is proved. □

Theorem 4. *For each $p \in \beta\mathbb{N} \setminus \mathbb{N}$, there is an invertible $T \in B(l_1)$ such that $T^{**}\delta_p$ is a Banach limit.*

Proof. For $n \geqslant 3$, let $T_n = (t_{ij})$ be the following $n \times n$ matrix:

$$\begin{pmatrix} 1/n & 1 & 1 & 1 & \cdots & 1 \\ 1/n & 1 & 0 & 0 & \cdots & 0 \\ 1/n & 0 & 1 & 0 & \cdots & 0 \\ 1/n & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/n & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Now $\sup_{1\leqslant j\leqslant n} \sum_{i=1}^n |t_{ij}| = 2$, and the inverse of T_n is:

$$T_n^{-1} = (s_{ij})_{n\times n} = \begin{pmatrix} -\frac{n}{n-2} & \frac{n}{n-2} & \frac{n}{n-2} & \frac{n}{n-2} & \cdots & \frac{n}{n-2} \\ \frac{1}{n-2} & 1-\frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & \cdots & -\frac{1}{n-2} \\ \frac{1}{n-2} & -\frac{1}{n-2} & 1-\frac{1}{n-2} & -\frac{1}{n-2} & \cdots & -\frac{1}{n-2} \\ \frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & 1-\frac{1}{n-2} & \cdots & -\frac{1}{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & \cdots & 1-\frac{1}{n-2} \end{pmatrix},$$

which satisfies $\sup_{1\leqslant j\leqslant n} \sum_{i=1}^n |s_{ij}| \leqslant 4$. Now define the infinite matrices

$$T = \begin{pmatrix} T_3 & 0 & 0 & \cdots \\ 0 & T_4 & 0 & \cdots \\ 0 & 0 & T_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ and } S = \begin{pmatrix} T_3^{-1} & 0 & 0 & \cdots \\ 0 & T_4^{-1} & 0 & \cdots \\ 0 & 0 & T_5^{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We see that both T and S are in $B(l_1)$, and that $S = T^{-1}$.

Let

$$n_k = \frac{(k+1)(k+2)}{2} - 2 = 1 + \sum_{i=1}^{k-1}(i+2),$$

and define $E = \{n_k \mid k \in \mathbb{N}\}$. For any $p \in \overline{E} \setminus E$, we claim that $T^{**}\delta_p$ is a Banach limit. To see this, we first observe that for $x = \{x_n\} \in l_\infty$, the n_k-th term of T^*x is $P_{n_k}T^*x = \sum_{i=0}^{k+1} x_{n_k+i}/(k+2)$, and $(T^{**}\delta_p)(x) = \int_{\beta\mathbb{N}} T^*x\, d\delta_p$ is a subsequential limit of the sequence $\{P_{n_k}T^*x\}$. From this it is obvious that $\|T^{**}\delta_p\| \leqslant 1$ and $(T^{**}\delta_p)(e) = 1$. For any $x = \{x_m\} \in bs$, $P_{n_k}T^*x = \sum_{i=0}^{k+1} x_{n_k+i}/(n+2) \to 0$ as $k \to \infty$, so $(T^{**}\delta_p)(x) = 0$. By Lemma G, $T^{**}\delta_p$ is a Banach limit.

For an arbitrary $p \in \beta\mathbb{N} \setminus \mathbb{N}$, take an infinite subset $P \subset \mathbb{N}$ with infinite complement in $\mathbb{N}$ such that $p \in \overline{P}$. Arrange the elements of P, $\mathbb{N} \setminus P$ and $\mathbb{N} \setminus E$ respectively in increasing order $p_1 < p_2 < \cdots$, $q_1 < q_2 < \cdots$ and $m_1 < m_2 < \cdots$, and define an infinite matrix $U = (u_{nk})$ by $u_{n_ip_i} = u_{m_iq_i} = 1$ for $i = 1, 2, \ldots$, and $u_{nk} = 0$ otherwise. Then U is in $B(l_1)$ with its inverse $V = (v_{nk})$ given by $v_{p_in_i} = u_{q_im_i} = 1$ for $k = 1, 2, \ldots$, and $v_{nk} = 0$ otherwise. Because for $F \subset \mathbb{N}$, $(U^{**}\delta_p)(\chi(F))$ is either 0 or 1, $U^{**}\delta_p$ is a Dirac measure. Clearly it is not on a point in $\mathbb{N}$. For $x = \{x_n\} \in l_\infty$, $P_{p_k}U^*(x) = x_{n_k}$. As a result, the support of $U^{**}\delta_p$ is in $\overline{E} \setminus E$. Applying what we got earlier in this proof, we know that $T^{**}U^{**}\delta_p$ is a Banach limit. □

Using Lemma D we get

Corollary. *Suppose* $p \in \beta\mathbb{N} \setminus \mathbb{N}$. *Then there is a Banach limit* $f \in l_1^{**}$ *such that* $\Gamma_p \cong \Gamma_f$ *and* $\Omega_p \cong \Omega_f$.

We used intersections of algebras associated with Banach limits in Theorem 3. Now we investigate the intersections in further detail. We first give a property of all measures μ such that $\mu(\mathbb{N}) = 0$, a condition which is satisfied by all Banach limits.

Lemma 2. *For any non-negative* μ *in* $M(\beta\mathbb{N}) = l_\infty^*$, *such that* $\mu(\mathbb{N}) = 0$, *there exists a subset* E *of* $\mathbb{N}$,

$$E = \bigcup_{i=1}^{\infty} \{n_i, n_i + 1, \dots, n_i + p(i) - 1\} \tag{4}$$

where $n_{i+1} \geqslant n_i + p(i)$ *and* $\lim_{i\to\infty} p(i) = \infty$, *such that* $\mu(\overline{E}) = 0$.

Proof. Let $E_0 = \mathbb{N} = \bigcup_{i=1}^{\infty} E_{i,0}$ where $E_{i,0} = \{i^2, i^2 + 1, \dots, i^2 + 2i\}$. Let

$$E_1' = \bigcup_{i=1}^{\infty} E_{2i-1,0} \text{ and } E_1'' = \bigcup_{i=1}^{\infty} E_{2i,0}.$$

So $E_0 = E_1' \cup E_1''$ and $E_1' \cap E_1'' = \varnothing$. Since disjoint subsets of $\mathbb{N}$ have disjoint closures in $\beta\mathbb{N}$ we see that

$$\mu(\overline{E_0}) = \mu(\overline{E_1'}) + \mu(\overline{E_1''}).$$

So we can choose E_1 to be either E_1' or E_1'' so that $\mu(\overline{E_1}) \leqslant (1/2)\mu(\overline{E_0})$.

Continuing in the same way we obtain a sequence of sets $\{E_n\}$ where $E_n = \bigcup_{i=1}^{\infty} E_{i,n}$, $\{E_{i,n+1}\}$ is a subsequence of $\{E_{i,n}\}$ and $\mu(\overline{E_n}) \to 0$ as $n \to \infty$. Let

$$E = \bigcup_{i=1}^{\infty} E_{i,i} = \bigcup_{i=1}^{\infty} \{n_i, n_i + 1, \dots, n_i + p(i) - 1\}.$$

Clearly $p(i) \to \infty$ as $n \to \infty$.

Now for each n, $E \setminus E_n$ is a finite subset of $\mathbb{N}$. Hence $\mu(\overline{E}) \leqslant \mu(\overline{E_n})$ so that $\mu(\overline{E}) = 0$. □

Let I stand for the identity operator. It was shown in Theorem 5 and Theorem 6 of [2] that $\bigcap\{\Gamma_w \mid w \in X^{**} \setminus \hat{X}\} = \langle I \rangle$ and $\bigcap\{\Omega_w \mid w \in X^{**} \setminus \hat{X}\} = \langle I \rangle \oplus W$ where W stands for the set of all weakly compact operators in $B(X)$. Using Lemma 2, we can characterize the operators in the intersection of Γ_f's for all $f \in BL$.

Theorem 5. *An operator* $T \in B(l_1)$ *is in* Γ_f *for all* $f \in BL$ *if and only if there is a complex number* ρ *and a matrix* $T_0 \in B(l_1)$, $T_0^* \in B(l_\infty, ac_0)$, *such that* $T = \rho I + T_0$. *where* ac_0 *is the space of sequences that are almost convergent to* 0. *That is,*

$$ac_0 = \{x = \{x_n\} \in l_\infty \mid f(x) = 0 \text{ for all } f \in BL\}.$$

Proof. The "if" part of the proof is straightforward. For the "only if" part, take any $T \in B(l_1)$ such that for $f \in BL$ we have $(T^{**}f)(x) = \rho_f f(x)$ for some $\rho_f \in \mathbb{C}$ and all $x \in l_\infty$. Now ρ_f is independent of f. Indeed, take any two

Banach limits a and b. By Lemma 2, there is a subset E of $\mathbb{N}$, given by (4) such that $(a+b)(\overline{E}) = 0$. Define $A : l_\infty \to l_\infty$ by $P_i Ax = (1/p(i)) \sum_{k=n_i}^{n_i+p(i)-1} x_k$ for $x = \{x_k\} \in l_\infty$. Take any extended limit α on l_∞ and let $h = \alpha A$. Then, since $p(i) \to \infty$ so that $Ax \in c_0$ when $x \in bs$, we see that $h(x) = 0$ when $x \in bs$. Lemma G shows that $h \in BL$. Also the support of μ_h is contained in $\overline{E}$. Now for any $x \in l_\infty$, remembering that $F = (1/2)(a+h) \in BL$, we have

$$\begin{aligned}(a+h)(T^*x) &= \rho_F(a+h)(x),\\ a(T^*x) &= \rho_a a(x),\\ h(T^*x) &= \rho_h h(x).\end{aligned}$$

and hence $\rho_F(a+h)(x) = \rho_a a(x) + \rho_h h(x)$. Knowing that the supports of μ_f and μ_h are disjoint, we see that $\rho_F = \rho_a = \rho_h$. Similarly $\rho_b = \rho_h$. Thus $\rho_a = \rho_b = \rho$. Let $T_0 = T - \rho I$. It is immediate that $T_0^* \in (l_\infty, ac_0)$. □

Theorem 6. *If $T \in \bigcap_{f \in BL} \Omega_f$, then there is a $\rho \in \mathbb{C}$ and a weak-star to norm continuous function $\phi : BL \to l_1$ such that*

(5) $\phi(\lambda f + (1-\lambda)g) = \lambda\phi(f) + (1-\lambda)\phi(g)$ *for all $\lambda \in [0, 1]$ and $f, g \in BL$,*

and $T^{**}f = \rho f + \widehat{\phi(f)}$ for all $f \in BL$. Moreover, $\phi(BL)$ is norm compact in l_1.

Proof. Suppose $T \in \bigcap_{f \in BL} \Omega_f$. Then for $f \in BL$, there exist $\rho_f \in \mathbb{C}$ and $x_f = \{x_k\} \in l_1$ such that $T^{**}f = \rho_f f + \widehat{x_f}$. We define $\phi(f) = x_f$. It is clear that ϕ satisfies (5). By making minor changes in the proof of Theorem 5, we see that ρ_f is again independent of f.

Now BL is a w^*-compact convex subset of l_∞^*. Also $\phi(BL)$ is a convex subset of l_1. For the w^*-norm continuity, we pick any net $\{f_\lambda \mid \lambda \in \Lambda\}$ in BL which converges to $f \in BL$ in the w^*-topology. Then for any $x \in l_\infty$,

$$x(\phi(f_\lambda)) = f_\lambda[(T^* - \rho I)(x)] \to f[(T^* - \rho I)(x)] = x(\phi(f)),$$

so ϕ is w^*-w continuous. As a result, $\phi(BL)$ is weakly compact. Eberlein-Šmulian's Theorem and Schur's Theorem together give us the norm compactness of $\phi(BL)$. By a standard result in general topology, the norm and the weak topologies on $\phi(BL)$ are the same. Thus ϕ is w^*-norm continuous. □

References

1. G. Bennett and N. J. Kalton, *Consistency theorems for almost convergence*, Tran. Amer. Math. Soc. **198** (1974), 23–43.
2. H. I. Brown, F. P. Cass and I. J. W. Robinson, *On isomorphisms between certain subalgebras of $B(X)$*, Studia Math. **63** (1978), 189–197.
3. H. I. Brown and T. Cho, *Subalgebras of $B(c)$*, Proc. Amer. Math. Soc. **40** (1973), 458–464.
4. H. I. Brown, D. R. Kerr and H. H. Stratton, *The structure of $B(c)$ and extensions of the concept of conull matrix*, Proc. Amer. Math. Soc. **22** (1969), 7–14.
5. F. P. Cass, *Subalgebras of $B(l_1)$ and the Stone-Čech compactification βN*, J. Funct. Anal. **32** (1979), 272–276.
6. ——, *On certain algebras of $B(X)$*, Analysis **6** (1986), 99–104.
7. W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–420.

8. ———, *Real and complex analysis*, McGraw-Hill, New York, 1987.

9. A. Wilansky, *Subalgebras of* $B(X)$, Proc. Amer. Math. Soc. **29** (1971), 355–360.

Department of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7

E-mail address: fcass@julian.uwo.ca